Extension of Matrices with Entries in H^∞ on Coverings of Riemann Surfaces of Finite Type

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Abstract

In the present paper continuing our work started in [Br1]-[Br5] we prove an extension theorem for matrices with entries in the algebra of bounded holomorphic functions defined on an unbranched covering of a Caratheodory hyperbolic Riemann surface of finite type.

1. Introduction.

Let X be a complex manifold and let $H^\infty(X)$ be the Banach algebra of bounded holomorphic functions on X equipped with the supremum norm. We assume that X is Caratheodory hyperbolic, that is, the functions in $H^\infty(X)$ separate the points of X. The maximal ideal space $\mathcal{M} = \mathcal{M}(H^\infty(X))$ is the set of all nonzero linear multiplicative functionals on $H^\infty(X)$. Since the norm of each $\phi \in \mathcal{M}$ is ≤ 1, $\mathcal{M}$ is a subset of the closed unit ball of the dual space $(H^\infty(X))^*$. It is a compact Hausdorff space in the Gelfand topology (i.e., in the weak $*$ topology induced by $(H^\infty(X))^*$). Also, there is a continuous embedding $i : X \hookrightarrow \mathcal{M}$ taking $x \in X$ to the evaluation homomorphism $f \mapsto f(x)$, $f \in H^\infty(X)$. The complement to the closure of $i(X)$ in $\mathcal{M}$ is called the *corona*. The *corona problem* is: given X to determine whether the corona is empty. For example, according to Carleson's celebrated Corona Theorem [C] this is true for X being the open unit disk in $\mathbb{C}$. (This was conjectured by Kakutani in 1941.) Also, there are non-planar Riemann surfaces for which the corona is non-trivial (see, e.g., [JM], [G], [BD], [L] and references therein). This is due to a structure that in a sense makes the surface seem higher dimensional. So there is a hope that the restriction to the Riemann sphere might prevent this obstacle. However, the general problem for planar domains is still open, as is the

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problem in several variables for the ball and polydisk. (In fact, there are no known examples of domains in $\mathbb{C}^n$, $n \geq 2$, without corona.) At present, the strongest corona theorem for planar domains is due to Moore [M]. It states that the corona is empty for any domain with boundary contained in the graph of a $C^{1+\epsilon}$-function. This result is an extension of an earlier result of Jones and Garnett [GJ] for a Denjoy domain (i.e., a domain with boundary contained in $\mathbb{R}$).

The corona problem can be reformulated as follows, see, e.g., [Ga]:

A collection $f_1, \dots, f_n$ of functions from $H^\infty(X)$ satisfies the *corona condition* if

$$\max_{1\leq j\leq n} |f_j(x)| \geq \delta > 0 \quad \text{for all} \quad x \in X. \tag{1.1}$$

The corona problem being solvable (i.e., the corona is empty) means that the Bezout equation

$$f_1 g_1 + \cdots + f_n g_n \equiv 1 \tag{1.2}$$

has a solution $g_1, \dots, g_n \in H^\infty(X)$ for any $f_1, \dots, f_n$ satisfying the corona condition. We refer to $\max_{1\leq j\leq n} ||g_j||_\infty$ as a "bound on the corona solutions". (Here $||\cdot||_\infty$ is the norm on $H^\infty(X)$.)

In [Br4, Theorem 1.1] using an L^2 cohomology technique we proved

Theorem 1.1 *Let $r: X \to Y$ be a connected unbranched covering of a Caratheodory hyperbolic Riemann surface of finite type Y (i.e., the fundamental group of Y is finitely generated). Then X is Caratheodory hyperbolic and for any $f_1, \dots, f_n \in H^\infty(X)$ satisfying (1.1) there are solutions $g_1, \dots, g_n \in H^\infty(X)$ of (1.2) with the bound* $\max_{1\leq j\leq n} ||g_j||_\infty \leq C(Y, n, \max_{1\leq j\leq n} ||f_j||_\infty, \delta)$.

This result extends the class of Riemann surfaces for which the corona theorem is valid (see also [Br1]). On the other hand, from the results of Lárusson [L] (sharpened in [Br3]) one obtains that the assumption of the Caratheodory hyperbolicity of Y cannot be removed. Specifically, for any integer $n \geq 2$ there are a compact Riemann surface S_n and its regular covering $r_n : \widetilde{S}_n \to S_n$ such that

(a) $\widetilde{S}_n$ is a complex submanifold of an open Euclidean ball $\mathbb{B}_n \subset \mathbb{C}^n$;

(b) the embedding $i : \widetilde{S}_n \hookrightarrow \mathbb{B}_n$ induces an isometry $i^* : H^\infty(\mathbb{B}_n) \to H^\infty(\widetilde{S}_n)$.

In particular, the maximal ideal spaces of $H^\infty(\widetilde{S}_n)$ and $H^\infty(\mathbb{B}_n)$ coincide.

The main result of our paper is the following noncommutative analog of the above theorem:

Theorem 1.2 *Let $r: X \to Y$ satisfy the assumptions of Theorem 1.1 and $A = (a_{ij})$ be an $n \times k$ matrix, $k < n$, with entries in $H^\infty(X)$. Assume that the family of minors of order k of A satisfies the corona condition. Then there is an $n \times n$ matrix $\widetilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} \in H^\infty(X)$, so that $\tilde{a}_{ij} = a_{ij}$ for $1 \leq j \leq k$, $1 \leq i \leq n$, and det $\widetilde{A} = 1$.*

Moreover, the corresponding norm of $\widetilde{A}$ is bounded by a constant depending on the norm of A, δ (from (1.1) for the family of minors of order k of A), n and Y only.

Previously, a similar result was proved for matrices with entries in $H^\infty(U)$ for domains $U \hookrightarrow X$ such that the embedding induces an injective homomorphism of the corresponding fundamental groups and $r(U) \subset\subset Y$, see [Br2, Theorem 1.1]. Its proof was based on a Forelli type theorem on projections in H^∞ (see [Br1]) and a Grauert type theorem for "holomorphic" vector bundles on maximal ideal spaces (which are not usual manifolds) of certain Banach algebras (see [Br2]) along with some ideas of Tolokonnikov [T] (see also this paper for further results and references on the extension problem for matrices with entries in different function algebras).

The remarkable class of Riemann surfaces X for which a Forelli type theorem and, hence, the corona theorem are valid was introduced by Jones and Marshall [JM]. The definition is in terms of an interpolating property for the critical points of the Green function on X. It is an interesting open question whether the result analogous to Theorem 1.2 is valid for such X.

2. Auxiliary Results.

2.1. For a set of indices Λ consider the family $X_\Lambda := \{X_\lambda\}_{\lambda\in\Lambda}$ where each X_λ is a connected unbranched covering of Y. By $r_\lambda := X_\lambda \to Y$ we denote the corresponding projection. Considering this family as the disjoint union of sets X_λ we introduce the natural complex structure on X_Λ. Thus $r_\Lambda : X_\Lambda \to Y$ is an unbranched covering of Y where $r_\Lambda|_{X_\lambda} := r_\lambda$.

We say that a function f on X_Λ belongs to $H^\infty(X_\Lambda)$ if $f|_{X_\lambda} \in H^\infty(X_\lambda)$, $\lambda \in \Lambda$, and $\sup_{\lambda\in\Lambda} ||f|_{X_\lambda}||_\infty < \infty$.

Proposition 2.1 *The corona theorem is valid for $H^\infty(X_\Lambda)$.*

Proof. Let $f_1, \dots, f_n \in H^\infty(X_\Lambda)$ satisfy the corona condition (1.1). We set $f_{j\lambda} := f_j|_{X_\lambda}$. Then each family $f_{1\lambda}, \dots, f_{n\lambda} \in H^\infty(X_\lambda)$ satisfies (1.1) with the same δ as for $f_1, \dots, f_n$. According to Theorem 1.1 there are functions $g_{1\lambda}, \dots, g_{n\lambda} \in H^\infty(X_\lambda)$ such that

$$f_{1\lambda}g_{1\lambda} + \cdots + f_{n\lambda}g_{n\lambda} \equiv 1$$

and

$$\max_{1\le j\le n} ||g_{j\lambda}||_\infty \le C(Y, n, \max_{1\le j\le n} ||f_j||_{H^\infty(X_\Lambda)}, \delta).$$

Let us define $g_1, \dots, g_n \in H^\infty(X_\Lambda)$ by the formulas

$$g_j|_{X_\lambda} := g_{j\lambda}.$$

Then $g_1 f_1 + \cdots + g_n f_n \equiv 1$. $\square$

Let $\mathcal{M}_\Lambda$ be the maximal ideal space of the Banach algebra $H^\infty(X_\Lambda)$. According to Theorem 1.1, $H^\infty(X_\Lambda)$ separates the points of X_Λ. Thus X_Λ can be regarded as a subset of $\mathcal{M}_\Lambda$. Now, by Proposition 2.1, X_Λ is dense in $\mathcal{M}_\Lambda$ in the Gelfand topology.

We will show that Theorem 1.2 follows directly from

Theorem 2.2 *Let $A = (a_{ij})$ be an $n \times k$ matrix, $k < n$, with entries in $H^\infty(X_\Lambda)$. Assume that the family of minors of order k of A satisfies the corona condition. Then there is an $n \times n$ matrix $\widetilde{A} = (\widetilde{a}_{ij})$, $\widetilde{a}_{ij} \in H^\infty(X_\Lambda)$, so that $\widetilde{a}_{ij} = a_{ij}$ for $1 \le j \le k$, $1 \le i \le n$, and det $\widetilde{A} = 1$.*

2.2. We recall some constructions and results presented in [Br2].

According to a construction of [Br2, section 2] the covering $r_\Lambda : X_\Lambda \to Y$ can be considered as a fibre bundle over Y with a discrete fibre F_Λ, where F_Λ is the disjoint union of the fibres F_λ of the coverings $r_\lambda : X_\lambda \to Y$, $\lambda \in \Lambda$. Let $l^\infty(F_\Lambda)$ be the Banach algebra of bounded complex-valued functions f on the discrete set F_Λ with pointwise multiplication and norm $||f|| = \sup_{x \in F_\Lambda} |f(x)|$. Let βF_Λ be the *Stone-Čech compactification* of F_Λ, i.e., the maximal ideal space of $l^\infty(F_\Lambda)$ equipped with the Gelfand topology. Then F_Λ is naturally embedded into βF_Λ as an open dense subset, and the topology on F_Λ induced by this embedding coincides with the original one, i.e., is discrete. Every function $f \in l^\infty(F_\Lambda)$ has a unique extension $\hat{f} \in C(\beta F_\Lambda)$. Further, any homeomorphism $\phi : F_\Lambda \to F_\Lambda$ determines an isometric isomorphism of Banach algebras $\phi^* : l^\infty(F_\Lambda) \to l^\infty(F_\Lambda)$. Therefore ϕ can be extended to a homeomorphism $\hat{\phi} : \beta F_\Lambda \to \beta F_\Lambda$. From here, taking closures in βF_λ of fibres of the bundle $r_\Lambda : X_\Lambda \to Y$, we obtain a fibre bundle $\hat{r}_\Lambda : E(Y, \beta F_\Lambda) \to Y$ with fibre βF_Λ so that X_Λ is an open dense subset of $E(Y, \beta F_\Lambda)$ (in fact, an open subbundle of $E(Y, \beta F_\Lambda)$) and $\hat{r}_\Lambda|_{X_\Lambda} = r_\Lambda$. Moreover, it was proved in [Br2, Proposition 2.1] that

(1) *for every $h \in H^\infty(X_\Lambda)$ there is a unique $\hat{h} \in C(E(Y, \beta F_\Lambda))$ such that $\hat{h}|_{X_\Lambda} = h$.*

Also, it was proved in [Br4, Theorem 1.5] that for every $x \in Y$ and every $\lambda \in \Lambda$ the sequence $r_\lambda^{-1}(x) \subset X_\lambda$ is interpolating for $H^\infty(X_\lambda)$ with the constant of interpolation bounded by a number depending on x and Y only. This immediately implies that

(2) *for each $f \in l^\infty(r_\Lambda^{-1}(x))$ there is a function $\tilde{f} \in H^\infty(X_\Lambda)$ such that $\tilde{f}|_{r_\Lambda^{-1}(x)} = f$.*

In particular, the continuous extension of the algebra $H^\infty(X_\Lambda)$ to $E(Y, \beta F_\Lambda)$ separates the points on $E(Y, \beta F_\Lambda)$. Thus $E(Y, \beta F_\Lambda)$ can be regarded as a dense subset of $\mathcal{M}_\Lambda$.

Let $(U_i)_{i \in I}$ be a countable cover of Y by compact subsets $U_i \subset Y$ homeomorphic to a closed ball in $\mathbb{R}^2$. Then by our construction $\widehat{U}_i := \hat{r}_\Lambda^{-1}(U_i)$ is homeomorphic to $U_i \times \beta F_\Lambda$. So, $E(Y, \beta F_\Lambda)$ is a countable union of compact subsets $\widehat{U}_i$. Since the covering dimension $dim\, \widehat{U}_i$ of $\widehat{U}_i$ is 2, $i \in I$, this implies (cf. [Br2, Proposition 4.1])

(3)
$$dim\, E(Y, \beta F_\Lambda) = 2.$$

Taking now an open countable cover of Y by relatively compact subsets homeomorphic to an open ball in $\mathbb{R}^2$ and the corresponding open cover of $E(Y, \beta F_\Lambda)$ by their preimages with respect to $\hat{r}_\Lambda$ we get

(4) *$E(Y, \beta F_\Lambda)$ is an open dense subset of $\mathcal{M}_\Lambda$, and the restriction of the Gelfand topology on $\mathcal{M}_\Lambda$ to $E(Y, \beta F_\Lambda)$ coincides with the topology of $E(Y, \beta F_\Lambda)$.*

2.3. Since Y is a Riemann surface of finite type, the theorem of Stout [St, Theorem 8.1] implies that there exist a compact Riemann surface R and a holomorphic embedding $\phi : Y \to R$ such that $R \setminus \phi(Y)$ consists of finitely many closed disks with analytic boundaries together with finitely many isolated points. Since Y is Caratheodory hyperbolic, the set of the disks in $R \setminus \phi(Y)$ is not empty. Also, without loss of generality we may and will assume that the set of isolated points in $R \setminus \phi(Y)$ is not empty, as well. (For otherwise, $\phi(Y)$ is a bordered Riemann surface and the required result follow from [Br2, Theorem 1.1].) We will naturally identify Y with $\phi(Y)$. Also, we set

$$R \setminus Y := \left(\bigsqcup_{1\le i\le k} \overline{D}_i\right) \bigcup \left(\bigcup_{1\le j\le l} \{x_j\}\right) \quad \text{and} \quad Z := Y \bigcup \left(\bigcup_{1\le j\le l} \{x_j\}\right) \tag{2.1}$$

where each D_i is biholomorphic to the open unit disk $\mathbb{D} \Subset \mathbb{C}$ and these biholomorphisms are extended to diffeomorphisms of the closures $\overline{D}_i \to \overline{\mathbb{D}}$. Then $Z \subset R$ is a bordered Riemann surface with a nonempty boundary. In particular, there is a bordered Riemann surface Z' containing $\overline{Z}$ such that $\overline{Z}$ is a deformation retract of Z'. We set

$$Y' := Z' \setminus \{x_1, \dots, x_l\}. \tag{2.2}$$

Then $Y \subset Y'$ and $\pi_1(Y) \cong \pi_1(Y')$ (here $\pi_1(M)$ stands for the fundamental group of M). This implies that for each $\lambda \in \Lambda$ there is a connected covering X'_λ of Y' such that X_λ is an open connected subset of X'_λ. Without loss of generality we denote the covering projection $X'_\lambda \to Y'$ by the same symbol r_λ (as for X_λ). Now, we define $X'_\Lambda := \{X'_\lambda\}_{\lambda\in\Lambda}$ so that X_Λ is an open subset of X'_Λ and $r_\Lambda : X'_\Lambda \to Y'$, $r_\Lambda|_{X'_\lambda} := r_\lambda$.

Further, similarly to the constructions of section 2.2 we determine the bundle $\hat{r}_\Lambda : E(Y', \beta F_\Lambda) \to Y'$ so that $E(Y, \beta F_\Lambda)$ is an open subbundle of $E(Y', \beta F_\Lambda)$. Then X'_Λ and $E(Y', \beta F_\Lambda)$ possess the properties similar to (1)-(3) for X_Λ and $E(Y, \beta F_\Lambda)$.

Let $cl(Y)$ denote the closure of Y in Y'. We set

$$E(cl(Y), \beta F_\Lambda) := \hat{r}_\Lambda^{-1}(cl(Y)).$$

Then we have

(5) *$dim\ E(cl(Y), \beta F_\Lambda) = 2$ and $E(Y, \beta F_\Lambda) \subset E(cl(Y), \beta F_\Lambda)$ is an open dense subset.*

2.4. By $H^\infty(E(Y, \beta F_\Lambda))$ we denote the extension of $H^\infty(X_\Lambda)$ to $E(Y, \beta F_\Lambda)$ described in section 2.2. We will use also the algebra $H^\infty(E(Y', \beta F_\Lambda))$ determined similarly (i.e., with Y and X_Λ substituted for Y' and X'_Λ).

Next, let us consider Banach subalgebras $\mathcal{A}_1$, $\mathcal{A}_2$ of $H^\infty(E(Y, \beta F_\Lambda))$ defined as follows.

$$\mathcal{A}_1 := \{\hat{r}_\Lambda^* f \in H^\infty(E(Y, \beta F_\Lambda)) \ : \ f \in H^\infty(Z')\}. \tag{2.3}$$

(Here $\hat{r}_\Lambda^* f$ is the pullback of f with respect to $\hat{r}_\Lambda$.)

To define $\mathcal{A}_2$ we choose a function $\phi \in H^\infty(Z')$ with the set of zeros $\{x_1, \dots, x_l\}$ so that each x_j is a zero of order 1 of ϕ. (Since $Z' \subset\subset R$ is a bordered Riemann

surface with a nonempty boundary, such a ϕ exists due to [Br2, Corollary 1.8].) Then $\mathcal{A}_2$ is the uniform closure of the algebra of functions $f \in H^\infty(E(Y, \beta F_\Lambda))$ of the form

$$f := g + ([\hat{r}_\Lambda^* \phi] \cdot h)|_{E(Y,\beta F_\Lambda)}, \ g \in \mathcal{A}_1, \ h \in H^\infty(E(Y', \beta F_\Lambda)). \tag{2.4}$$

By the definition $\mathcal{A}_2$ separates the points of $E(Y, \beta F_\Lambda)$ (because $H^\infty(E(Y', \beta F_\Lambda))$ separates the points of $E(Y', \beta F_\Lambda)$ and $\hat{r}_\Lambda^* \phi$ is nonzero on the fibres of $\hat{r}_\Lambda$).

Clearly, we have embeddings

$$\mathcal{A}_1 \stackrel{i_1}{\longrightarrow} \mathcal{A}_2 \stackrel{i_2}{\longrightarrow} H^\infty(E(Y, \beta F_\Lambda)). \tag{2.5}$$

The transpose maps to these embeddings determine continuous surjective maps

$$\mathcal{M}_\Lambda \stackrel{i_2^*}{\longrightarrow} M_2 \stackrel{i_1^*}{\longrightarrow} M_1 \tag{2.6}$$

where M_2 is the closure in the Gelfand topology of the image of $E(Y, \beta F_\Lambda)$ in the maximal ideal space of $\mathcal{A}_2$, and M_1 is the closure in the Gelfand topology of the image of $E(Y, \beta F_\Lambda)$ in the maximal ideal space of $\mathcal{A}_1$. (Here we used that the closure in the Gelfand topology of $E(Y, \beta F_\Lambda) \subset \mathcal{M}_\Lambda$ is $\mathcal{M}_\Lambda$, see Proposition 2.1.)

By the definition, $M_1 = \overline{Z}$ and $E(cl(Y), \beta F_\Lambda) \subset M_2$ (see section 2.3). Moreover, the restriction of i_2^* to $E(Y, \beta F_\Lambda)$ is the identity map and the restriction of i_1^* to $E(cl(Y), \beta F_\Lambda)$ can be naturally identified with $\hat{r}_\Lambda$ so that $(i_1^*)^{-1}(cl(Y)) = \hat{r}_\Lambda^{-1}(cl(Y)) = E(cl(Y), \beta F_\Lambda)$. Now, we prove

Lemma 2.3 *For each $x_j \in M_1$ the compact set $(i_1^*)^{-1}(x_j)$ consists of a single point (which we naturally identify with x_j), $1 \le j \le l$.*

Proof. Let $\{\xi_{1,\alpha}\}, \{\xi_{2,\alpha}\} \subset E(Y, \beta F_\Lambda)$ be nets converging to points $\xi_1, \xi_2 \in (i_1^*)^{-1}(x_j)$. Then for f from (2.4) and $i = 1, 2$ we have

$$f(\xi_i) = \lim_\alpha f(\xi_{i,\alpha}) = \lim_\alpha (g(\xi_{i,\alpha}) + (\hat{r}_\Lambda^* \phi)(\xi_{i,\alpha}) \cdot h(\xi_{i,\alpha})) := g(x_j). \tag{2.7}$$

(We used here that the nets $\{i_1(\xi_{1,\alpha})\}, \{i_1(\xi_{2,\alpha})\} \subset M_1$ converge to x_j.)

This implies that $\xi_1 = \xi_2$. $\square$

Corollary 2.4

$$dim\, M_2 = 2.$$

Proof. According to Lemma 2.3 and property (5) of section 2.3, M_2 is the disjoint union of zero-dimensional sets $\{x_j\}$, $1 \le j \le l$, and the two-dimensional set $E(cl(Y), \beta F_\Lambda)$. Hence $dim\, M_2 = 2$, see, e.g., [N, Chapter 2, Theorem 9-11]. $\square$

2.5. We fix coordinate neighbourhoods $N_j \subset\subset Z$ (see (2.1)) biholomorphic to $\mathbb{D}$ of points x_j, $1 \le j \le l$, and a bordered Riemann surface $S \subset Y$ such that $N_i \cap N_j = \emptyset$ for $i \ne j$, each $Y \cap N_j$ does not contain x_j and is biholomorphic to an

annulus and $\mathcal{U} := S \cup (\cup_{1\le j\le l} N_j^*)$ is an open cover of Y. Here $N_j^* := N_j \setminus \{x_j\}$ is biholomorphic to $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$. We set

$$N_{j\Lambda}^* := r_\Lambda^{-1}(N_j^*), \quad 1 \le j \le l, \quad \text{and} \quad S_\Lambda := r_\Lambda^{-1}(S). \tag{2.8}$$

Let $V \subset X_\Lambda$ be either one of $N_{j\Lambda}^*$ or S_Λ. By $H^\infty(V)$ we denote the Banach algebra of bounded holomorphic functions on V defined as in section 2.1 for X_Λ. Further, we set

$$\hat{N}_j := (i_2^* \circ i_1^*)^{-1}(N_j), \quad 1 \le j \le l, \quad \hat{S} := (i_2^* \circ i_1^*)^{-1}(S \cup \partial Z) \tag{2.9}$$

Here ∂Z is the boundary of the bordered Riemann surface Z that can be regarded as the "outer boundary" of S.

By the definition $\hat{N}_j$, $1 \le j \le l$, and $\hat{S}$ are open subsets of $\mathcal{M}_\Lambda$ forming a cover of this space. The main fact used in the proof of Theorem 1.1 is

Proposition 2.5 *Assume that $f \in H^\infty(V)$ where V is either one of $N_{j\Lambda}^*$ or S_Λ. Then f admits a continuous extension $\hat{f}$ to $\hat{V}$ where $\hat{V}$ stands for the corresponding $\hat{N}_j$ or $\hat{S}$.*

Proof. First, we will prove the result for $V = N_{j\Lambda}^*$.

Let ρ_j be a C^∞-function on R equal to 1 in a neighbourhood of x_j with $supp(\rho) \subset\subset N_j$. We set

$$f_1 := (r_\Lambda^* \rho_j) \cdot f. \tag{2.10}$$

Then f_1 can be considered as a C^∞-function on X'_Λ (defined in section 2.3). Further, we introduce a $(0,1)$-form on X'_Λ by the formula

$$\omega := \frac{\overline{\partial} f_1}{\rho_\Lambda^* \phi}. \tag{2.11}$$

The definition is correct because $\overline{\partial} f_1$ equals 0 on $\rho_\Lambda^{-1}(O)$ for some neighbourhood O of x_j and on $X'_\Lambda \setminus N_{j\Lambda}^*$, and $\rho_\Lambda^* \phi \ne 0$ on $N_{j\Lambda}^*$. Thus ω is a $\overline{\partial}$-closed 1-form on X'_Λ. Consider the form

$$\omega_\lambda := \omega|_{X'_\lambda} \quad \text{on} \quad X'_\lambda.$$

Let us assume that Z' is equipped with a hermitian metric $h_{Z'}$ with the associated $(1,1)$-form $\omega_{Z'}$. Then we equip X'_λ with the hermitian metric $h_{X'_\lambda}$ induced by the pullback $r_\lambda^* \omega_{Z'}$ of $\omega_{Z'}$ to X'_λ. Now, if η is a smooth $(0,1)$-form on X'_λ, by $|\eta|_z$, $z \in X'_\lambda$, we denote the norm of η at z defined by the hermitian metric $h_{X'_\lambda}^*$ on the fibres of the cotangent bundle $T^* X'_\lambda$ on X'_λ.

Next, since $f \in H^\infty(N_{j\Lambda}^*)$ and $r_\Lambda(supp(\omega)) =: K \subset\subset Y'$, see (2.2), one easily obtains from (2.11) that

$$||\omega|| := \sup_{\lambda \in \Lambda} \left\{ \sup_{z \in X_\lambda} |\omega_\lambda|_z \right\} < \infty \tag{2.12}$$

From here by [Br4, Theorem 1.6] we obtain that the equation $\overline{\partial} g_\lambda = \omega_\lambda$ has a smooth bounded solution g_λ on X'_λ such that

$$||g_\lambda||_{L^\infty} := \sup_{z\in X'_\lambda} |g_\lambda(z)| \le C||\omega|| \tag{2.13}$$

with C depending on K, Z' and $h_{Z'}$ only.

We define bounded functions g and f_2 on X'_Λ by the formulas

$$g|_{X'_\lambda} := g_\lambda, \quad \lambda \in \Lambda, \quad f_2 := (\rho_\Lambda^* \phi)\cdot g. \tag{2.14}$$

Then we have

$$(a)\ \ \overline{\partial} f_2 = \overline{\partial} f_1 \quad \text{on} \quad X'_\Lambda \quad \text{and} \quad (b)\ \ \lim_\alpha f_2(\xi_\alpha) = 0 \tag{2.15}$$

for each net $\{\xi_\alpha\} \subset X'_\Lambda$ such that $\{r_\Lambda(\xi_\alpha)\} \subset Y'$ is a net converging to any x_s, $1 \le s \le l$. In particular,

$$f_3 := f_1 - f_2 \in H^\infty(X'_\Lambda). \tag{2.16}$$

Thus f_3 admits a continuous extension $\hat f_3$ to $\mathcal{M}_\Lambda$.

Let us prove now that

(*) *f_2 admits a continuous extension $\hat f_2$ to $\mathcal{M}_\Lambda$.*

Indeed, by the definition of f and $r_\Lambda^*\rho_j$, the function f_1 defined by (2.10) has a continuous extension to $E(Y', \beta F_\Lambda)$, see [Br2, Proposition 2.1]. Thus, $f_2 := f_1 - f_3$ admits a continuous extension to $E(Y', \beta F_\Lambda)$, as well. (We denote this extension also by f_2.) Now if $\{\xi_\alpha\} \subset E(Y', \beta F_\Lambda)$ is a net converging to a point $\xi \in M_2$ (see (2.6)) such that $i_1^*(\xi) = x_s$ for some $1 \le s \le l$, then from (2.15) (b) we get

$$\lim_\alpha f_2(\xi_\alpha) = 0.$$

Since $(i_1^*)^{-1}(x_s) = x_s$, the latter implies that the function $\tilde f_2$ equals 0 at each x_s and f_2 on $E(cl(Y), \beta F_\Lambda)$ is continuous on M_2. Therefore the function $\hat f_2 := i_2^* \tilde f_2$ is continuous on $\mathcal{M}_\Lambda$. Since the restriction of i_2^* to $E(Y, \beta F_\Lambda)$ is the identity map, $\hat f_2$ is a continuous extension of f_2. This proves (*).

From (2.16) and (*) we obtain that f_1 admits a continuous extension $\hat f_1$ to $\mathcal{M}_\Lambda$. Now, $\hat N_j$ from (2.9) is the union of $\hat r_\Lambda^{-1}(N_j^*) \subset E(Y, \beta F_\Lambda)$ and $(i_2^* \circ i_1^*)^{-1}(O)$ where $O \subset\subset N_j$ is a neighbourhood of x_j such that $\rho_j \equiv 1$ on O. The function f admits a continuous extension $\tilde f$ on $\hat r_\Lambda^{-1}(N_j^*)$, see [Br2, Proposition 2.1], and $f = f_1$ on $r_\Lambda^{-1}(O)$. Thus the function $\hat f$ defined by

$$\hat f := \hat f_1 \quad \text{on} \quad (i_2^* \circ i_1^*)^{-1}(O) \quad \text{and} \quad \hat f := \tilde f \quad \text{on} \quad \hat r_\Lambda^{-1}(N_j^*)$$

is the required continuous extension of f to $\hat N_j$.

Finally, in the case $V = S_\Lambda$ we choose a C^∞-function ρ on R equals 0 on $Y \setminus S$ and 1 on $R \setminus Z$ with $supp(d\rho) \subset\subset S$. Then repeating the above arguments with ρ_j substituted for ρ we obtain the proof of the proposition in this case. We leave the details to the readers. □

3. Proof of Theorem 1.2.

3.1. Proof of Theorem 2.2. Let $A = (a_{ij})$ be an $n \times k$ matrix, $k < n$, with entries in $H^\infty(X_\Lambda)$. Assume that the family of minors of order k of A satisfies the corona condition (1.1). Due to the corona theorem for $H^\infty(X_\Lambda)$, see Proposition 2.1, we can extend A continuously to $\mathcal{M}_\Lambda$ such that the family of minors of order k of the extended matrix $\hat{A} = (\hat{a}_{ij})$ satisfies (1.1) on $\mathcal{M}_\Lambda$ with the same δ as for A. Next, according to [L, Theorem 3], to prove the theorem it suffices to find an $n \times n$ matrix $B = (b_{ij})$, $b_{ij} \in C(\mathcal{M}_\Lambda)$, so that $b_{ij} = \hat{a}_{ij}$ for $1 \le j \le k$, $1 \le i \le n$, and $det\, B = 1$.

Note that the matrix $\hat{A}$ determines a trivial subbundle ξ of complex rank k in the trivial vector bundle $\theta^n := \mathcal{M}_\Lambda \times \mathbb{C}^n$ on $\mathcal{M}_\Lambda$. Let η be an additional to ξ subbundle of θ^n, i.e., $\xi \oplus \eta = \theta^n$. We will prove that η is topologically trivial. Then a trivialization $s_1, s_2, \dots, s_{n-k} \in C(\mathcal{M}_\Lambda, \eta)$ (given by global continuous sections of η) will determine the required continuous extension B of the matrix $\hat{A}$.

Let us prove first that $\hat{A}$ can be extended to an invertible matrix on each $\hat{N}_j$ and $\hat{S}$, see (2.9).

Lemma 3.1 *Let $\hat{V}$ be either one of $\hat{N}_j$ or $\hat{S}$. Then for $\hat{A}|_{\hat{V}}$ there is an $n \times n$ matrix $B_{\hat{V}} = (b_{ij;\hat{V}})$, $b_{ij;\hat{V}} \in C(\hat{V})$, so that $b_{ij;\hat{V}} = \hat{a}_{ij}|_{\hat{V}}$ for $1 \le j \le k$, $1 \le i \le n$, and $det\, B_{\hat{V}} = 1$. Moreover, $B_{\hat{V}}|_V$ has entries in $H^\infty(V)$ where $V := \hat{V} \cap X_\Lambda$.*

Proof. First assume that $\hat{V} = \hat{N}_j$ so that $V = N^*_{j\Lambda}$ is an unbranched covering of $N^*_j \cong \mathbb{D}^*$, see (2.8). Then by the definition $N^*_{j\Lambda} = \{N^*_{j\lambda}\}_{\lambda \in \Lambda}$ where each $N^*_{j\lambda} := N^*_{j\Lambda} \cap X_\lambda$ is an unbranched covering of N^*_j consisting of at most countably many connected components. Thus each $N^*_{j\lambda}$ is biholomorphic to $\sqcup_{k \in K_\lambda} W_{j\lambda;k}$, $K_\lambda \subset \mathbb{N}$, where each $W_{j\lambda;k}$ is either $\mathbb{D}$ or $\mathbb{D}^*$. Now, $A|_{W_{j\lambda;k}}$ satisfies conditions of Theorem 1.1 with the same δ as for A. According to the main result of Tolokonnikov [T] for H^∞-matrices on $\mathbb{D}$, there is a matrix $B_{j\lambda;k}$ with entries in $H^\infty(W_{j\lambda;k})$ which extends $A|_{W_{j\lambda;k}}$ in the sense of Theorem 1.1 and such that $det\, B_{j\lambda;k} = 1$ and

$$\sup_{j,\lambda,k} ||B_{j\lambda;k}|| \le C \tag{3.1}$$

where C depends on the norm of A on X_Λ, δ and n. (Here for a matrix $C = (c_{ij})$ with entries in $H^\infty(O)$ we set $||C|| := \max_{i,j} ||c_{ij}||_{H^\infty(O)}$.) In particular, (3.1) implies that the matrix $B_{j\Lambda}$ on $N^*_{j\Lambda}$ defined by

$$B_{j\Lambda}|_{W_{j\lambda;k}} = B_{j\lambda;k}, \quad 1 \le j \le l,\ k \in K_\lambda,\ \lambda \in \Lambda,$$

has entries in $H^\infty(N^*_{j\Lambda})$, extends $A|_{N^*_{j\Lambda}}$ and $det\, B_{j\Lambda} = 1$. According to Proposition 2.5, $B_{j\Lambda}$ is extended to a continuous matrix $B_{\hat{N}_j}$ on $\hat{N}_j$. This matrix extends $\hat{A}|_{\hat{N}_j}$ and satisfies the required conditions of the lemma.

Consider now the case $\hat{V} = \hat{S}$ so that $V = S_\Lambda$ is an unbranched covering of S, see (2.8). In this case we apply similar to the above arguments where instead of the result of [T] we use [Br2, Theorem 1.1] applied to the coverings $S_\lambda := S_\Lambda \cap X_\lambda$ of a bordered Riemann surface S. Then we obtain a matrix B_Λ on S_Λ with entries in

$H^\infty(S_\Lambda)$ which extends $A|_{S_\Lambda}$ and such that $det\, B_\Lambda = 1$. Applying again Proposition 2.5, we extend B_Λ continuously to $\hat{S}$ so that the extended matrix $B_{\hat{S}}$ satisfies the required conditions of the lemma. $\Box$

Let ξ_q be the quotient bundle of θ^n with respect to the subbundle ξ. By the definition ξ_q is isomorphic (in the category of continuous bundles on $\mathcal{M}_\Lambda$) to η. Thus it suffices to prove that ξ_q is topologically trivial.

Now, Lemma 3.1 implies straightforwardly that $\xi_q|_{\hat{V}}$ is topologically trivial for $\hat{V}$ being either one of $\hat{N}_j$ or $\hat{S}$. In particular, ξ_q is defined by a 1-cocycle defined on the open cover $\{\hat{N}_1, \dots, \hat{N}_l, \hat{S}\}$ of $\mathcal{M}_\Lambda$ (see, e.g., [H] for the general theory of vector bundles). Since by the definition $\hat{N}_i \cap \hat{N}_j = \emptyset$ for $i \neq j$, this cocycle consists of continuous matrix-functions

$$C_i \in C(\hat{N}_i \cap \hat{S}, GL_{n-k}(\mathbb{C})), \quad 1 \le i \le l.$$

Set now $\tilde{N}_j := (i_1^*)^{-1}(N_j)$, $\tilde{S} := (i_1^*)^{-1}(S)$, see (2.6). Then $\{\tilde{N}_1, \dots, \tilde{N}_l, \tilde{S}\}$ is an open cover of M_2. Moreover, the map $i_2 : \mathcal{M}_\Lambda \to M_2$ is identity on each $\hat{N}_i \cap \hat{S}$, see section 2.4. Therefore each C_i can be regarded as a matrix-function on $\tilde{N}_i \cap \tilde{S}$. In particular, these functions determine a complex vector bundle $\tilde{\xi}_q$ of rank $n-k$ on M_2 so that

$$i_2^* \tilde{\xi}_q = \xi_q. \tag{3.2}$$

Since $dim\, M_2 = 2$, see Corollary 2.4, the bundle $\tilde{\xi}_q$ is isomorphic to $\theta_{M_2}^{n-k-1} \oplus \theta$ where $\theta_{M_2}^{n-k-1} := M_2 \times \mathbb{C}^{n-k-1}$ is the trivial bundle and θ is a vector bundle of complex rank 1, see, e.g., [Br5, Lemma 2.8]. This and (3.2) imply that $\xi_q \cong \eta$ is isomorphic to $\theta^{n-k-1} \oplus i_2^*\theta$ where $\theta^{n-k-1} = \mathcal{M}_\Lambda \times \mathbb{C}^{n-k-1}$ is the trivial bundle. Now, for the first Chern classes (which are additive with respect to the operation of the direct sum of bundles) we have the following identity

$$0 = c_1(\theta^n) = c_1(\xi) + c_1(\eta) = c_1(\theta^{n-k-1} \oplus i_2^*\theta) = c_1(i_2^*\theta). \tag{3.3}$$

We used here that Chern classes of trivial bundles are zeros.

Equality (3.3) shows that the first Chern class of the complex rank 1 vector bundle $i_2^*\theta$ is zero. Thus this bundle is topologically trivial (see, e.g., [H]). Combining this fact with the above isomorphism for η we get $\eta \cong \theta^{n-k} := \mathcal{M}_\Lambda \times \mathbb{C}^{n-k}$.

This completes the proof of Theorem 2.2. $\Box$

3.2. Proof of Theorem 1.1. Let us define $\Lambda_{Y;M,\delta}$ as the set of all possible couples (A, X) where X is a connected covering of Y and A is an $n \times k$ matrix on X satisfying conditions of Theorem 1.1 with a fixed δ in the corona condition (1.1) for the family of minors of order k and such that $||A|| \le M$. For $\Lambda := \Lambda_{Y;M,\delta}$ we consider the $n \times k$ matrix $\mathcal{A}$ with entries in $H^\infty(X_\Lambda)$ defined as follows

$$\mathcal{A}|_{X_\lambda} := A, \quad \lambda = (A, X) \in \Lambda, \quad X_\lambda := X. \tag{3.4}$$

Then clearly $\mathcal{A}$ satisfies conditions of Theorem 2.2 on X_Λ. According to this theorem there is an $n \times n$ matrix $\widetilde{\mathcal{A}}$ with entries in $H^\infty(X_\Lambda)$ and with $det\, \widetilde{\mathcal{A}} = 1$ that extends $\mathcal{A}$. For $\lambda = (A, X) \in \Lambda$ we set

$$\widetilde{A} := \widetilde{\mathcal{A}}|_X.$$

Then $\widetilde{A}$ extends A and det $\widetilde{A} = 1$, and $||\widetilde{A}|| \leq C(||A||, \delta, M, Y)$.

The proof of Theorem 1.1 is complete. $\square$

References

[BD] D. E. Barrett, and J. Diller, A new construction of Riemann surfaces with corona. J. Geom. Anal. **8** (1998), 341-347.

[Br1] A. Brudnyi, Projections in the space H^∞ and the Corona Theorem for coverings of bordered Riemann surfaces. Ark. Mat. **42** (2004), no. 1, 31-59.

[Br2] A. Brudnyi, Grauert and Lax-Halmos type theorems and extension of matrices with entries in H^∞. J. Funct. Anal. **206** (2004), 87-108.

[Br3] A. Brudnyi, A uniqueness property for H^∞ on coverings of projective manifolds. Michigan Math. J. **51** (2003), no. 3, 503-507.

[Br4] A. Brudnyi, Corona Theorem for H^∞ on Coverings of Riemann Surfaces of Finite Type. Michigan Math. J., to appear.

[Br5] A. Brudnyi, Matrix-valued corona theorem for multiply connected domains. Indiana Univ. Math. J., **49** (4) (2000), 1405-1410.

[C] L. Carleson, Interpolation of bounded analytic functions and the corona problem. Ann. of Math. **76** (1962), 547-559.

[Ga] J. B. Garnett, *Bounded analytic functions*. Academic Press, 1981.

[G] T. W. Gamelin, *Uniform algebras and Jensen measures*. London Math. Soc. Lecture Notes Series **32**. Cambridge Univ. Press, Cambridge-New York, 1978.

[GJ] J. B. Garnett, and P. W. Jones, The corona theorem for Denjoy domains. Acta Math. **155** (1985), 27–40.

[H] D. Husemoller, *Fibre bundles*. Springer Verlag. New York, 1966.

[JM] P. W. Jones, and D. Marshall, Critical points of Green's functions, harmonic measure and the corona theorem. Ark. Mat. **23** (1985), 281-314.

[N] K Nagami, *Dimension theory*. Academic Press. New York, 1970.

[L] F. Lárusson, Holomorphic functions of slow growth on nested covering spaces of compact manifolds. Canad. J. Math. **52** (2000), 982-998.

[Li] V. Lin, Holomorphic fibering and multivalued functions of elements of a Banach algebra. Funct. Anal. and its Appl. English translation, **7** (2) (1973), 122-128.

[M] C. N. Moore, The corona theorem for domains whose boundary lies in a smooth curve. Proc. Amer. Math. Soc. **100** (1987), no. 2, 266-270.

[St] E. L. Stout, Bounded holomorphic functions on finite Riemann surfaces. Trans. Amer. Math. Soc. **120** (1965), 255-285.

[T] V. Tolokonnikov, Extension problem to an invertible matrix. Proc. Amer. Math. Soc. **117** (1993), no. 4, 1023-1030.